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ON St. VENANT FLEXURE INCLUDING COUPLE STRESSES

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Introduction. We consider the St. Venant flexure problem for beams of narrow rectangular cross section, that is under the assumption of plane stress, for two reasons. The first of these is that it is possible to give an exact solution in closed form for this problem including significant effects of couple stresses. The second reason is that this problem may be considered as a special case of the problem of deriving two-dimensional shell theory from three-dimensional elasticity theory in the iterative manner which has been presented for the general case in September 1967 in Kopenhagen at the Second Symposium on Shell Theory of the International Union of Theoretical and Applied Mechanics (IUTAM).

Formulation of the problem. Appropriate differential equations are three equilibrium equations

$$\sigma_{xx,x} + \sigma_{yx,y} = 0, \quad \sigma_{xy,x} + \sigma_{yy,y} = 0, \tag{1a, b}$$

$$\tau_{x,x} + \tau_{y,y} + \sigma_{xy} - \sigma_{yx} = 0 \tag{1c}$$

three compatibility equations (*)

$$e_{xx,y} - e_{yx,x} + k_x = 0, \quad e_{xy,y} - e_{yy,x} + k_y = 0, \quad k_{x,y} - k_{y,x} = 0 \tag{2a, b, c}$$

and six stress strain relations which are here been taken in the form

$$E_x e_{xx} = \sigma_{xx} - \nu_x \sigma_{yy}, \quad E_y e_{yy} = \sigma_{yy} - \nu_y \sigma_{xx} \tag{3a, b}$$

$$2G_x e_{xy} = \sigma_{xy}, \quad 2G_y e_{yx} = \sigma_{yx} \tag{3c, d}$$

$$c^2 \Gamma_x k_x = \tau_x, \quad c^2 \Gamma_y k_y = \tau_y \tag{3e, f}$$

with $(\nu_x / E_x = \nu_y / E_y)$.

The system (1) to (3) is to be solved in the rectangular region $|y| \leqslant c, |x| \leqslant L$ subject to boundary conditions

$$y = \pm c, \quad \sigma_{yx} = \sigma_{yy} = \tau_y = 0 \tag{4}$$

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*) which are a consequence of strain displacement relations

$$e_{xx} = u_{,x}, \ e_{yy} = v_{,y}, \ k_x = \psi_{,x}, \ k_y = \psi_{,y}, \ e_{xy} = v_{,x} - \psi, \ e_{yx} = v_{,y} + \psi$$

$$x = \pm L, \qquad \int_{-c}^{c} (\sigma_{xx}, \sigma_{xy}, y\sigma_{xx} - \tau_x)\, dy = (0, Q, \pm QL) \tag{5}$$

In prescribing conditions for $x = \pm L$ in integrated form, we allow for a solution by means of the semi-inverse procedure.

Reduction to second-order differential equation. We stipulate in advance that σ_{xx} and τ_x are linear in x, that σ_{xy}, σ_{yx} and τ_y are independent of x and that σ_{yy} vanishes identically, and write (6a, b, c, d, e, f)

$$\sigma_{xx} = xS_x, \ \tau_x = xT_x, \ \sigma_{yy} = 0, \ \sigma_{xy} = S_{xy}, \ \sigma_{yx} = S_{yx}, \ \tau_y = T_y$$

Introduction of (6) into equations (1) to (3), with primes indicating differentiation with respect to y, leaves the following system

$$S_x + S_{yx}' = 0, \ T_x + T_y' + S_{xy} - S_{yx} = 0 \tag{7a, b}$$

$$\left(\frac{S_x}{E_x}\right)' + \frac{T_x}{c^2\Gamma_x} = 0, \quad \left(\frac{S_{xy}}{2G_x}\right)' + \frac{\nu_x S_x}{E_x} + \frac{T_y}{c^2\Gamma_y} = 0, \quad \left(\frac{T_x}{c^2\Gamma_x}\right)' = 0 \tag{8a, b, c}$$

From (8c) and (8a) follows

$$\frac{T_x}{c^2\Gamma_x} = K, \qquad \frac{S_x}{E_x} = -K(y - y_0) \tag{9a, b}$$

where K and y_0 are constants of integration.

From (8a) and (4) follows further

$$S_{yx} = K \int_{-c}^{y} E_x(\eta - y_0)\, d\eta, \qquad y_0 \int_{-c}^{c} E_x\, dy = \int_{-c}^{c} yE_x\, dy \tag{10a, b}$$

Equation (7b) is written in the form

$$S_{xy} = S_{yx} - T_x - T_y' \tag{11}$$

and introduced into (8b) so as to leave the second order differential equation

$$\left(\frac{T_y'}{2G_x}\right)' - \frac{T_y}{c^2\Gamma_y} = \left(\frac{S_{yx} - T_x}{2G_x}\right)' + \frac{\nu_x S_x}{E_x} \tag{12}$$

together with the boundary conditions $T_y(\pm c) = 0$.

The constant K in Eqs.(9)–(12) is expressed in terms of the applied forces Q by means of the relation

$$-K \int_{-c}^{c} [c^2\Gamma_x + y(y - y_0)E_x]\, dy = Q \tag{13}$$

The geometrical meaning of K follows upon writing $v_{,xx} = e_{xy,x} - k_x = \tau_x / c^2\Gamma_x = Kx$.

Explicit solution for uniform cross section beam. With all E, ν, G and Γ independent of y we have first from (10b) and (13)

$$y_0 = 0, \qquad K = \frac{-Q}{2\Gamma_x c^3 + {}^2/_3 E_x c^3} \tag{14a, b}$$

From (9), (10) and (6) follow further as expressions for σ_{xx}, σ_{xy} and τ_x

$$\sigma_{xx} = \frac{E_x Qxy}{2\Gamma_x c^3 + {}^2/_3 E_x c^3}, \qquad \sigma_{yx} = \frac{{}^1/_2 E_x Q\,(c^2 - y^2)}{2\Gamma_x c^3 + {}^2/_3 E_x c^3} \tag{15a, b}$$

$$\tau_x = \frac{-\Gamma_x Qc^2 x}{2\Gamma_x c^3 + {}^2/_3 E_x c^3} \tag{15c}$$

The differential equation (12) reduces to

$$T_y'' - \frac{2G_x}{\Gamma_y c^2} T_y = (E_x - 2\nu_x G_x) K y \tag{16}$$

from which, with $T_y(\pm c) = 0$,

$$T_y = -\frac{E_x - 2\nu_x G_x}{\lambda^2} K\left(y - c\frac{\operatorname{sh}\lambda y}{\operatorname{sh}\lambda c}\right), \qquad \lambda^2 = \frac{2G_x}{\Gamma_y c^2} \tag{17}$$

and then, in view of (6f)

$$\tau_y \quad \left(\frac{1}{2}\frac{E_x}{G_x} - \nu_x\right)\frac{\Gamma_y Q c^3}{2\Gamma_x c^3 + {}^2/_3 E_x c^3}\left(\frac{y}{c} - \frac{\operatorname{sh}\lambda y}{\operatorname{sh}\lambda c}\right) \tag{18}$$

Finally, Eqs. (11) and (6d) give for the distribution of transverse shears (19)

$$\sigma_{xy} = \frac{Qc^2}{2\Gamma_x c^3 + {}^2/_3 E_x c^3}\left[\frac{E_x}{2}\left(1 - \frac{y^2}{c^2}\right) + \Gamma_x - \left(\frac{1}{2}\frac{E_x}{G_x} - \nu_x\right)\Gamma_y\left(1 - \frac{\lambda c \operatorname{ch}\lambda y}{\operatorname{sh}\lambda c}\right)\right]$$

Insofar as the significance of Eqs. (15), (18) and (19) is concerned, we may assume that x is of order L and that $c/L \ll 1$.

The case $\Gamma_y = 0$. The assumption that the medium can support couple stresses between transverse fibers but that there can be no couple stresses between longitudinal fibers, reduces Eq. (19) to

$$\sigma_{xy} = \frac{Q\,[{}^1/_2 E_x(c^2 - y^2) + \Gamma_x c^2]}{2\Gamma_x c^3 + {}^2/_3 E_x c^3} \tag{20}$$

and leaves equations (15) unchanged.

We see that as long as $\Gamma_x \ll E_x$ we have $\sigma_{xy} \approx \sigma_{yx}$, and both these stresses are small compared to σ_{xx}.

When Γ_x is the same order as E_x we still have that σ_{xy} and σ_{yx} are small compared to σ_{xx}. The difference between σ_{xy} and σ_{yx}, however, is now of the same order of magnitude as these stresses themselves.

Finally, when $E_x \ll \Gamma_x$, then σ_{xy} is effectively uniform across the thickness of the beam and, moreover, as large or larger than the longitudinal normal stress σ_{xx}. At the same time, the contribution of the force stresses σ_{xx} to the section moment Qx is now small compared to the contribution of the couple stresses τ_x to this moment.

The case $\Gamma_y = O(\Gamma_x)$. For this case we are, additionally, interested in the magnitude of τ_y relative to τ_x and in the relative magnitudes of the vatious components of strain.

We will first consider this question subject to the restrictive assumptions that $E_x/G_x = O(1)$ and $\nu_x = O(1)$ It is then readily seen from (18) and (15c) that $\tau_y = O(\tau_x c/L)$ and that σ_{xy} is either small compared to σ_{xx}, which is the case as long as $\Gamma_x/E_x = O(1)$, or σ_{xy} is effectively given by the Γ_x-term in (19).

For a comparison of strains we reintroduce the quantity K of Eq. (14b) and write

$$e_{xx} = -Kxy, \qquad e_{yy} = K\nu_x xy, \qquad e_{yx} = K\frac{E_x}{G_y}\frac{c^2 - y^2}{4} \tag{20a, b, c}$$

$$e_{xy} = K\left[\frac{E_x}{G_x}\frac{c^2 - y^2}{4} + \frac{\Gamma_x c^2}{2G_x} - \left(\frac{E}{2G_x} - \nu_x\right)\frac{\Gamma_y c^2}{G_x}\left(1 - \lambda c\frac{\operatorname{ch}\lambda y}{\operatorname{sh}\lambda c}\right)\right] \tag{20d}$$

$$k_x = -Kx, \qquad k_y = K\left(\frac{E}{2G_x} - \nu_x\right)c\left(\frac{y}{c} - \frac{\operatorname{sh}\lambda y}{\operatorname{sh}\lambda c}\right) \tag{21a, b}$$

We now have that k_y is small compared to k_x, for all possible values of λ. Furthermore, assuming that $E_x/G_y = O(1)$, we have that e_{yx} is small compared to e_{xx}.

The normal strain e_{yy} is of the same order of magnitude as e_{xx}, by way of the effect of Poisson's ratio, as expected.

In appraising e_{xy} we have that while the first term with E_x/G_x is small compared to e_{xx}, the term with Γ_x/G_x does not necessarily have this property. In order to see the effect of the term with Γ_y/G_x we must consider the entire range of values of $\lambda = \sqrt{2G_x/\Gamma_y}/c$.

Writing
$$\frac{\Gamma_y}{G_x}\left(1-\lambda c\,\frac{\operatorname{ch}\lambda y}{\operatorname{sh}\lambda c}\right)=\frac{2}{(\lambda c)^2}\left(1-\lambda c\,\frac{\operatorname{ch}\lambda y}{\operatorname{sh}\lambda c}\right) \tag{22}$$
it is seen that this term makes a contribution which is at the most of the same order of magnitude as the first term inside the brackets in (20d) and consequently is also small compared to e_{xx}.

Altogether we have then that except for terms small of higher order the nature of the state of strain in the beam is effectively as if
$$e_{xx}=-Kxy,\quad e_{yy}=K\nu_x xy,\quad e_{yx}=0 \tag{23a, b, c}$$
$$e_{xy}=K\frac{\Gamma_x c^2}{2G_x},\quad k_x=-Kx,\quad k_y=0 \tag{23e, f, g}$$

Removal of the restriction $E_x/G=O(1)$ means that transverse shear deformation may have a first order effect on the state of strain in the beam. Evidently, in contrast to the conclusions implied by Eqs. (23), it is now possible that k_y is of the same order of magnitude as k_x and that e_{yx} is of the same order of magnitude as e_{xx}.

Integro-differential equation form of the problem. We deduce from the equilibrium equations (1) and (4) the relations
$$\sigma_{yx}=-\int_{-c}^{y}\sigma_{xx,x}\,d\eta,\qquad \sigma_{yy}=-\int_{-c}^{y}\sigma_{xy,x}\,d\eta \tag{24a, b}$$
$$\tau_y=-\int_{-c}^{y}\left(\tau_{x,x}+\sigma_{xy}+\int_{-c}^{y}\sigma_{xx,x}\,d\zeta\right)d\eta \tag{24c}$$
and
$$\left(\int_{-c}^{c}\sigma_{xx}\,d\eta\right)_{,x}=0,\qquad \left(\int_{-c}^{c}\sigma_{xy}\,d\eta\right)_{,x}=0 \tag{25a, b}$$
$$\left[\int_{-c}^{c}(\tau_x-\eta\sigma_{xx})\,d\eta\right]_{,x}+\int_{-c}^{c}\sigma_{xy}\,d\eta=0 \tag{25c}$$

We deduce from the compatibility equations (2) the relations
$$k_x=K_x+\int_0^y k_{y,x}\,d\eta,\qquad e_{xy}=\gamma_x-\int_0^y k_y\,d\eta+\int_0^y e_{yy,x}\,d\eta \tag{26a, b}$$
$$e_{xx}=\varepsilon_x-K_x y+\int_0^y\left(e_{yx}-\int_0^{\eta}k_y\,d\zeta\right)_{,x}d\eta \tag{26c}$$
where K_x, λ_x and ε_x are functions of integration which are independent of the thickness coordinate y.

Introduction of (24) and (26) into the stress strain relations (3) leaves as a system of integro-differential equations

$$E_x\left[\varepsilon_x - K_x y + \int_0^y \left(e_{yx} - \int_0^\eta k_y\, d\zeta\right)_{,x} d\eta\right] = \sigma_{xx} + \nu_x \int_{-c}^{y} \sigma_{xy,x}\, d\eta \tag{27a}$$

$$E_y e_{yy} = -\int_{-c}^{y} \sigma_{xy,x}\, d\eta - \nu_y \sigma_{xx} \tag{27b}$$

$$2G_x\left[\gamma_x - \int_0^y k_y\, d\eta + \int_0^y e_{yy,x}\, d\eta\right] = \sigma_{xy} \tag{27c}$$

$$2G_y e_{yx} = -\int_{-c}^{y} \sigma_{xx,x}\, d\eta, \qquad c^2\Gamma_x \left[K_x + \int_0^y k_{y,x}\, d\eta\right] = \tau_x \tag{27d, e}$$

$$c^2\Gamma_y k_y = -\int_{-c}^{y} \left(\tau_{x,x} + \sigma_{xy} + \int_{-c}^{\eta} \sigma_{xx,x} d\zeta\right) d\eta \tag{27f}$$

Equations (27) may be considered as a system of six partial integro-differential equations for six dependent variables σ_{xx}, σ_{xy}, τ_x, e_{yx}, e_{yy}, k_y with ε_x, K_x and γ_x being parameters to be determined subsequently through use of the integrated equilibrium conditions (25).

The iterative procedure. Precondition for the possibility of solving the system (27) by means of an iterative procedure is the limitation to solutions with smallest characteristic length L, large compared to the thickness dimension c. By smallest characteristic length we mean that (smallest) length over which significant changes of the dependent variables occur, i. e.

$$e_{yx,x} = O\left(\frac{e_{yx}}{L}\right), \qquad \sigma_{xy,x} = O\left(\frac{\sigma_{xy}}{L}\right), \text{ etc.} \tag{28a, b}$$

and

$$\int_0^y e_{yx,x}\, d\eta = O\left(\frac{c}{L} e_{yx}\right), \text{ etc.} \tag{28c}$$

In applying the concept of this length L, its existence is assumed, the consequences of its existence are developed and finally it is verified that the properties of the solution which is obtained sre such as to make the initial assumption consistent.

The simplest type of iterative solution of the system (27) which may be considered is one in which all x-derivatives are assumed to be small of higher order. This, however, turns out to be excessively restrictive. A more satisfactory choice for the first step of the iterative procedure is to begin with the system

$$E_x\, [\varepsilon_x^{(1)} - K_x^{(1)} y] = \sigma_{xx}^{(1)}, \; E_y e_{yy}^{(1)} = -\nu_y \sigma_{xx}^{(1)} \tag{29a, b}$$

$$2G_x\left[\gamma_x^{(1)} - \int_0^y k_y^{(1)}\, d\eta - \int_0^y e_{yy,x}^{(1)}\, d\eta\right] = \sigma_{xy}^{(1)} \tag{29c}$$

$$2G_y e_{yx}^{(1)} = -\int_{-c}^{y} \sigma_{xx,x}^{(1)}\, d\eta, \qquad c^2\Gamma_x K_x^{(1)} = \tau_x^{(1)} \tag{29d, e}$$

$$c^2\Gamma_y k_y^{(1)} = -\int_{-c}^{y}\left(\tau_{x,x}^{(1)} + \sigma_{xy}^{(1)} + \int_{-c}^{\eta}\sigma_{xx,x}^{(1)}\,d\zeta\right)d\eta \tag{29f}$$

together with equations (25) for the quantities $\sigma_{xx}^{(1)}$, etc.

We now have $\sigma_{xx}^{(1)}$ and $\tau_x^{(1)}$ directly from (29a) and (29e) and then $e_{yy}^{(1)}$ and $e_{yx}^{(1)}$ from (29b) and (29d). It remains to evaluate (29c) and (29f) so as to obtain $\sigma_{xy}^{(1)}$ and $k_y^{(1)}$. For this purpose we first obtain from (25c) and (29f) an ordinary differential equation for $k_y^{(1)}$

$$\begin{aligned} 2G_x\left[\gamma_x^{(1)} - \int_0^y k_y^{(1)}\,d\eta + \int_0^y (\nu_y/E_y)\,\sigma_{xx,x}^{(1)}\,d\eta\right] &= \\ = -c^2\left(\Gamma_y k_y^{(1)}\right)' - \tau_{x,x}^{(1)} - \int_{-c}^{y}\sigma_{xx,x}^{(1)}\,d\eta & \end{aligned} \tag{30}$$

in which $\sigma_{xx}^{(1)}$ and $\tau_x^{(1)}$ are explicitly given as functions of y by (29a) and (29e).

The second order differential equation for $\int_0^y k_y^{(1)}\,d\eta$

is to be solved subject to the boundary conditions $\Gamma_y k_y^{(1)} = 0$ for $y = \pm c$. Having $k_y^{(1)}$ we then also have $\sigma_{xy}^{(1)}$ from (29c) or (29f) in terms of $\gamma_x^{(1)}$, $\varepsilon_x^{(1)}$ and $K_x^{(1)}$. Finally, equations (25) are used as ordinary differential equations for the determination of these three quantities as a function of x. For the relatively simple beam problem which is being considered, these differential equations may be reduced in order through integration, to read

$$\int_{-c}^{c}\sigma_{xx}^{(1)}\,d\eta = N, \qquad \int_{-c}^{c}\sigma_{xy}^{(1)}\,d\eta = Q, \qquad \int_{-c}^{c}\left(\tau_x^{(1)} - \eta\sigma_{xx}^{(1)}\right)d\eta = M + Qx \tag{31a, b, c}$$

In this problem of St. Venant flexure as defined through the boundary conditions (5) is the case $N = 0$ and $M = 0$, with $N \neq 0$ and $M \neq 0$ being associated with superposed problems of pure bending and stretching.

Having the first step-solution $\sigma_{xx}^{(1)}$, etc., determination of a second step solution $\sigma_{xx}^{(2)}$, etc. requires the solution of the system

$$E_x\left[\varepsilon_x^{(2)} - K_x^{(2)}y + \int_0^y\left(e_{yx}^{(1)} - \int_0^\eta k_y^{(1)}\,d\zeta\right)_{,x}d\eta\right] = \sigma_{xx}^{(2)} - \nu_x\sigma_{yy}^{(1)} \tag{32a}$$

$$E_y e_{yy}^{(2)} = \sigma_{yy}^{(1)} - \nu_y\sigma_{xx}^{(2)} \tag{32b}$$

$$2G_x\left[\gamma_x^{(2)} - \int_0^y k_y^{(2)}\,d\eta + \int_0^y e_{yy,x}^{(2)}\,d\eta\right] = \sigma_{xy}^{(2)} \tag{32c}$$

$$2G_y e_{yx}^{(2)} = -\int_{-c}^{y}\sigma_{xx,x}^{(2)}\,d\eta, \qquad c^2\Gamma_x\left[K_x^{(2)} + \int_0^y k_{y,x}^{(1)}\,d\eta\right] = \tau_x^{(2)} \tag{32d, e}$$

$$c^2\Gamma_y k_y^{(2)} = -\int_{-c}^{y}\left(\tau_{x,x}^{(2)} + \sigma_{xy}^{(2)} + \int_{-c}^{\eta}\sigma_{xx,x}^{(2)}\,d\zeta\right)d\eta \tag{32f}$$

together with equations (31) for $\sigma_{xx}^{(2)}$, $\sigma_{yx}^{(2)}$ and $\tau_x^{(2)}$.

Conditions for the validity of the procedure follow from a consideration of the system (27) and (29) in the form

$$\frac{c}{L} e_{yx}^{(1)} = O(\varepsilon_x^{(1)}), \quad \frac{c}{L} k_y^{(1)} = O(K_x^{(1)}), \quad \frac{c}{L} \sigma_{xy}^{(1)} = O(\sigma_{xx}^{(1)}) \qquad (33)$$

Equations (33) will be found to imply the requirement of certain stipulations concerning admissible relative orders of magnitudes of the coefficient functions E, G, Γ and ν. At the same time, more stringent types of stipulations (such as for example that $E_x = O(G)$) would allow us the use of a first-step equation system without some of the terms retained in (29) (without affecting the validity of the results obtained through use of equations (29) as they stand).

Finally, we note the following property of the equations of the iterative procedure above. For the case that all elasticity coefficient functions E, G, ν and Γ are independent of the coordinate x, that is for the case corresponding to equations (6) to (13), the terms omitted in going from equations (27) to equations (29) happen to be those x-derivative terms which vanish in the exact solution. As a consequence, for this special case the results of the first step of the iterative procedure, in the maximally complete form (29), would not be modified by the subsequent steps of the procedure.

ON THE STABILITY OF THREE-DIMENSIONAL ELASTIC BODIES

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The static stability of a three-dimensional elastic body with small subcritical strains is considered. Because of the assumption of smallness of the subcritical strains the results obtained below are applicable to the investigation of the stability of elastic bodies fabricated from a metal and from stiff bonded plastics. These results are also necessary for the latter since bonded plastics have low shear stiffness, hence application of the applied theories sometimes results in large errors in determining the critical forces.

Special attention is paid to obtaining general solutions of the static stability equations of a three-dimensional body compressed along the Ox_3 axis by stress resultants of intensity q, and along the Ox_1 and Ox_2 axes by stress resultants of intensity p. In the particular case of $p = 0$, solutions of a similar form [1] permitted the investigation of the stability of cylindrical shells [2] and bars [3]. The first members of the asymptotic expansions of the magnitudes of the critical force, which agree with the value of the critical force obtained with the aid of the Kirchhoff-Love hypothesis, were calculated in [2] and [3].

General solutions in invariant form are constructed below, which permit the investigation of the stability of hollow cylindrical shells, and of shells with a filler, of bars, of plates both single and multilayered subjected to the loadings mentioned above. As an illustration, the stability of rectangular and circular plates under multilateral compression is considered, where the boundary conditions are satisfied approximately in the integral sense.

Let us consider the static stability of a three-dimensional body with small subcritical strains compressed by stress resultants of intensity q along the x_3 axis and by stress